

Anyonic FRT construction ¹

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Abstract

The Faddeev-Reshetikhin-Takhtajan method to construct matrix bialgebras from non-singular solutions of the quantum Yang-Baxter equation is extended to the anyonic or \mathbf{Z}_n -graded case. The resulting anyonic quantum matrices are braided groups in which the braiding is given by a phase factor.

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1 Introduction

Braided groups [1] are a generalization of both quantum groups and supergroups. They have a coproduct $\underline{\Delta}$ which is a homomorphism not to the usual commutative tensor product but to the braided tensor product. The two factors in the braided tensor product no longer commute but instead have braid statistics described by a braiding Ψ . A superquantum group has Ψ given by super-transposition (with a factor ± 1). The next simplest and first truly braided case is the notion of an anyonic quantum group [2] [3] [4] where Ψ has a factor $e^{2\pi i/n}$ in place of ± 1 .

This note is a short announcement of [5] where the construction of quantum matrices of this anyonic type is achieved. We recall that the Faddeev-Reshetikhin-Takhtajan (FRT) construction [6] associates a quantum matrix bialgebra $A(R)$ to each non-singular solution R of the quantum Yang-Baxter equation. Its matrix of generators $\{t^i_j\}$ is like the co-ordinates on the set of matrices M_n , i.e. these bialgebras are of ‘function algebra type’. There is not exactly in [6] a corresponding construction for the dual bialgebras of ‘enveloping algebra type’ but there is a dually-paired bialgebra $\widetilde{U(R)}$ with matrix generators l^\pm which can generally be quotiented down to something like an enveloping algebra quantum group. A nice thing about this approach is that the bialgebras and also the duality pairing have as input data only the R -matrix. Many R -matrices are known so we have a rich and powerful construction. The generalization of this FRT or R -matrix setup to the case of super-quantum matrices is also known and is equally rich [7] [8]. The input data in this case is a solution of the *super Yang-Baxter equation*, which is still older [9]. Our paper [8] gave an abstract picture of this super FRT procedure in terms of a theory of transmutation which can be used to construct super and braided quantum groups from ordinary ones. Its extension now in the anyonic matrix case is one of our motivations.

By an anyonic or \mathbf{Z}_n -graded bialgebra we understand more precisely that the algebra structure is \mathbf{Z}_n -graded and that two copies of the algebra are multiplied according to the rule

$$(a \otimes b) (c \otimes d) = e^{\frac{2\pi i}{n} p(b)p(c)} (ac \otimes bd) \quad (1)$$

where a, b, c, d are homogeneous elements in the algebra whose grading is given by the function $p(\cdot)$ with values in the set $\mathbf{Z}_n = \{0, 1, \dots, n-1 \pmod{n}\}$. The coalgebra is defined with coproduct $\underline{\Delta}$ which is an algebra homomorphism from one copy to two copies

multiplied in this way. We will obtain anyonic bialgebras $\underline{A}(\underline{R})$ with a matrix $\{a^i_j\}$ of generators starting from the \mathbf{Z}_n -graded Yang-Baxter equation, which we also introduce. The $n = 1$ case recovers the usual or ungraded quantum matrices, while the $n = 2$ case recovers the super-quantum matrices mentioned above. Also in [5] is the theory of anyonic quantum groups $\underline{U}(\underline{R})$ of enveloping algebra type, as well as anyonic quantum planes $\underline{V}(\underline{R})$ on which they act. We shall outline some of this further theory too in the last section of this note.

It should be stressed that the jump from $n = 2$, the *super* case, to $n > 2$ gradation represents a drastic discontinuity due to the fact that many properties that hold for the former do not always hold in an analogous manner for the latter. This is because for $n > 2$ the permutation group generated by transpositions Ψ must truly be replaced by the Artin braid group. One of the resulting complications is manifested in the following example: a \mathbf{Z}_n -graded algebra that satisfies the relation $ab = e^{\frac{2\pi i}{n}p(a)p(b)}ba$ for all elements a and b in the algebra is called super-commutative when $n = 2$ but when $n > 2$ the relation is inconsistent; it is obvious then that whatever the definition of anyonic-commutativity should be in the $n > 2$ case it cannot be given by such formula. Another example is provided by the Jacobi identity for Lie superalgebras. Since there is such an identity it is not unreasonable to wonder about its \mathbf{Z}_n -graded analogue. We think that these unknowns can be best answered by first building up a good supply of anyonic quantum groups. Anyonic quantum matrices of function algebra type should give us some insight into the correct notion of anyonic-commutative algebras, while their duals of enveloping algebra type should give us some insight into the correct notion of anyonic Lie algebras, in both cases as the ‘classical limit’ of suitable \mathbf{Z}_n -graded examples. The notion of ‘quantization’ or q -deformation is independent of the notion of anyonic or other braiding. This can be regarded as one of the general motivations behind [5].

Other applications of the graded FRT bialgebras would be to physical systems where the relation (1) is involved. Of course this is not the case of a 4-dimensional world where all the operators associated to real particles satisfy either commuting or anticommuting relations (they are operators in an ungraded or in a super algebra) depending whether they describe bosons or fermions but it might be a 2-dimensional one where some anyonic behaviour is described. This is also the reason why we refer to \mathbf{Z}_n -gradation when $n > 2$ as *anyonic*. Moreover, in any dimension the process of q -deformation (for example for the purpose

of q -regularization or as a quantum correction to geometry) also spoils the spin-statistics theorem and indeed does typically induce braid statistics. For example, the dilaton generator in some approaches to the q -deformed Poincaré group reflects a residual \mathbf{Z} -grading (with corresponding braid statistics) induced by the q -deformation.

To conclude this introduction, we mention that the ultimate justification for studying the anyonic bialgebras derived from the modified FRT construction that we expose is that this is a construction simple to manipulate and without inconsistencies. There is no doubt then that these anyonic structures can teach something to us.

2 Anyonic Yang-Baxter equation

As we have mentioned in the introduction the consideration of gradation brings some modifications with respect to the ungraded situation. The first modification is that the ordinary Yang-Baxter equation is substituted by its graded analogue. In this case of \mathbf{Z}_n -gradation that we are studying its graded analogue is referred as the *anyonic Yang-Baxter equation* and it is introduced in the present section.

Let us consider a vector space V of finite dimension d with the direct sum decomposition $V = \bigoplus_{\alpha=0}^{n-1} V_\alpha$, $n \geq 2$ such that there is defined a function with values in the group \mathbf{Z}_n given by $p(x) \equiv \alpha \pmod{n}$ when x lies in V_α . This direct sum defines a \mathbf{Z}_n -gradation on V and any vector $v \in V$ is uniquely determined by its *homogeneous component of degree α* $v_\alpha \in V_\alpha$ by way of the expansion $v = \bigoplus_{\alpha=0}^{n-1} v_\alpha$. The gradation extends to the vector space $V \otimes V = \bigoplus_{\alpha,\beta=0}^{n-1} V_\alpha \otimes V_\beta$ which also becomes \mathbf{Z}_n -graded with the degree of any homogeneous element $v_\alpha \otimes v_\beta$ in $V_\alpha \otimes V_\beta$ defined by $p(v_\alpha \otimes v_\beta) \equiv p(v_\alpha) + p(v_\beta) = \alpha + \beta \pmod{n}$.

Let $\{e_i \otimes e_j\}, i, j \in I$ be a linear basis for $V \otimes V$ with the indexing set $I \equiv \{1, \dots, d\}$ and assume that all vectors e_i are homogeneous of degree $p(e_i) = p(i)$ where we use the convention that Latin indices run over the set I to avoid confusion with Greek indices used to denote degree.

Under these assumptions we say that an invertible matrix \underline{R} in $\text{End}(V \otimes V)$ is an *anyonic R -matrix* if

$$\underline{R}_{c \ d}^a \ ^b = 0 \quad \text{when} \quad p(a) + p(b) - p(c) - p(d) \not\equiv 0 \pmod{n}, \quad (2)$$

which we call the ‘null degree condition’ and

$$e^{\frac{2\pi i}{n} p(e)[p(f)-p(c)]} \underline{R}_{e \ f}^b \ ^a \underline{R}_{k \ c}^i \ ^f \underline{R}_{j \ d}^k \ ^e = e^{\frac{2\pi i}{n} p(r)[p(s)-p(a)]} \underline{R}_{p \ r}^i \ ^b \underline{R}_{j \ s}^p \ ^a \underline{R}_{d \ c}^r \ ^s, \quad (3)$$

which we call the *anyonic Yang-Baxter equation*. A sum over repeated indices is understood. Obviously the ungraded Yang-Baxter equation is obtained from this one with the particularization $n = 1$. Likewise, the super Yang-Baxter equation of [9] is recovered with the choice $n = 2$.

Note that for $n = 1$ the condition (2) is empty since any integer is zero mod 1. This is why we do not see it in the usual FRT procedure. For $n = 2$ the condition is equivalent to saying that \underline{R} is an even matrix since the degree of each matrix element is given by $p(\underline{R}_{c\ d}^a\ ^b) = p(a) + p(b) - p(c) - p(d)$ for all n .

There is no big loss of generality in adopting this null degree assumption and it has, on the contrary, many advantages for the simplifications that it introduces in the calculations. Neither this condition (2) nor the relation (3) change under a basis transformation in V of the type $e'_j = S_j^i e_i$ with $p(e'_i) = p(e_i)$ for all i and thus both conditions are invariant under the transformation $\underline{R} \rightarrow (S \otimes S)^{-1} \underline{R} (S \otimes S)$ where S is any non-singular $d^2 \times d^2$ matrix of null degree.

The following properties characterize anyonic R-matrices:

1. A solution to the *braid relation*

$$\check{R}_{e\ f}^a\ ^b \check{R}_{k\ c}^f\ ^i \check{R}_{j\ d}^e\ ^k = \check{R}_{p\ r}^b\ ^i \check{R}_{j\ s}^a\ ^p \check{R}_{d\ c}^s\ ^r \quad (4)$$

can be obtained from every solution of (2)–(3) through the transformation $\check{R} = \underline{P} \underline{R}$ where \underline{P} denotes the *anyonic permutation operator* in the \mathbf{Z}_n -graded vector space $V \otimes V$ defined by its action $\underline{P}(a \otimes b) = e^{\frac{2\pi i}{n} p(a)p(b)} b \otimes a$ on homogeneous vectors a and b in V or by its components on any linear basis of V (we assume basis transformation in V as those mentioned above) $\underline{P}_{c\ d}^a\ ^b = e^{\frac{2\pi i}{n} p(a)p(b)} \delta_d^a \delta_c^b$. A simple calculation indicates that this operator represents the n th-root of the permutation operator P with action $P(a \otimes b) = b \otimes a$ when n is odd and the n th-root of the identity operator in $V \otimes V$ when n is even.

Note that the equality (4) does not contain any degree label $p(\cdot)$ which shows that the gradation does not alter the braid relation.

2. There exists a relation between the solutions of (2)–(3) and the solutions of the ungraded Yang-Baxter equation given by the formula

$$R_{c\ d}^a\ ^b = e^{\frac{2\pi i}{n} p(a)p(b)} \underline{R}_{c\ d}^a\ ^b, \quad (5)$$

where the non underlined variables refer to the ungraded situation.

3. The matrices \underline{R}^\pm defined by $\underline{R}^+ \equiv \underline{P}\underline{R}\underline{P}$ and $\underline{R}^- \equiv \underline{R}^{-1}$ satisfy both the same equation

$$e^{\frac{2\pi i}{n}p(e)[p(c)-p(f)]} \underline{R}^{\pm b}{}_e{}^a{}_f \underline{R}^{\pm i}{}_k{}^f{}_c \underline{R}^{\pm k}{}_j{}^e{}_d = e^{\frac{2\pi i}{n}p(r)[p(a)-p(s)]} \underline{R}^{\pm i}{}_p{}^b{}_r \underline{R}^{\pm p}{}_j{}^a{}_s \underline{R}^{\pm r}{}_d{}^s{}_c, \quad (6)$$

but this does *not* coincide with equation (3) except in the ungraded or super case.

The proof of all these properties uses the null degree condition satisfied by the matrix elements of \underline{R} . Properties 1 and 2 are very helpful to find solutions to the anyonic Yang-Baxter equation since, for instance, many known solutions of the ordinary Yang-Baxter relation are transformable into solutions of its anyonic counterpart. Of course one should be careful and be sure that in the transformation the requisite of null parity of \underline{R} holds. An application of property 3 will be mentioned in the last section.

3 Anyonic FRT construction

We know from [6] that to any non-singular $d^2 \times d^2$ matrix solution R of the ordinary Yang-Baxter equation (we dont use underlines when we refer to the ungraded case) it is possible to associate a matrix bialgebra $A(R)$ generated by $\mathbb{1}$ and $\{t^i_j\}$ $i, j = 1, \dots, d$ with algebra relations

$$R^a{}_f{}^b{}_e t^f{}_c t^e{}_d = t^b{}_r t^a{}_s R^s{}_c{}^r{}_d, \quad (7)$$

and coalgebra given by

$$\Delta t^i_j = \sum_{k=1}^d t^i_k \otimes t^k_j, \quad \varepsilon(t^i_j) = \delta^i_j. \quad (8)$$

As we have explained in the introduction the association is still possible in the \mathbf{Z}_n -graded case but to a solution of (2)–(3) instead. Thus we define the associated anyonic bialgebra, $\underline{A}(\underline{R})$, as generated by $\mathbb{1}$ and $\{a^i_j\}$ $i, j = 1, \dots, d$ with algebra and coalgebra relations

$$e^{\frac{2\pi i}{n}[p(a)p(b)+p(c)p(e)]} \underline{R}^a{}_f{}^b{}_e a^f{}_c a^e{}_d = e^{\frac{2\pi i}{n}[p(c)p(d)+p(r)p(a)]} a^b{}_r a^a{}_s \underline{R}^s{}_c{}^r{}_d, \quad (9)$$

$$\underline{\Delta} a^i_j = \sum_{k=1}^d a^i_k \otimes a^k_j, \quad \underline{\varepsilon}(a^i_j) = \delta^i_j. \quad (10)$$

The algebra structure of this new bialgebra is \mathbf{Z}_n -graded with the generators of the bialgebra defined as homogeneous elements of degree $p(a^i_j) \equiv p(i) - p(j) \pmod{n}$ and as a graded

algebra the product of any two homogeneous elements x, y in the algebra is homogeneous and moreover satisfies $p(xy) = p(x) + p(y)$.

Notice that it is not possible to get rid of the phases in equation (9) with the transformation (5). This is an indication of the intrinsic anyonic character of the algebra that we are introducing and shows that the graded algebra is *not* the ungraded one (7) with the substitution of the ordinary Yang-Baxter equation solution by its graded analogue. Rather, one is obtained from the other by a process of transmutation, which changes the algebra product.

Regarding the coalgebra structure (10) $\underline{\triangle} a^i_j$ satisfies the relation (9) provided that two copies of the algebra multiply according to the law

$$(a \otimes b)(c \otimes d) = e^{\frac{2\pi i}{n} p(b)p(c)} (ac \otimes bd)$$

for all homogeneous elements a, b, c, d in $\underline{A}(\underline{R})$. This law corresponds to the particular case $\Psi(b \otimes c) = e^{\frac{2\pi i}{n} p(b)p(c)} c \otimes b$ of a more general concept of *braiding* Ψ [1]. Although it is not our purpose to review braided groups here, we see that the construction presented in this note modifies the FRT approach to also include the anyonic braiding.

We now give a non-trivial example of our construction, for $n = 3$. Our starting point is the solution of the ordinary Yang-Baxter equation given by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \quad (11)$$

taken from section 4 of [10] and written in the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, \dots, e_3 \otimes e_3\}$ of $V \otimes V$. Here $q = e^{2\pi i/3}$. Let us assume that V is a \mathbf{Z}_3 -graded vector space with degree of each basis vector given by $p(e_i) = i - 1$ for all $i = 1, 2, 3$ (note that the obvious inequality $d \geq n$ holds then). Under these assumptions this solution is associated by (5) with the solution of the anyonic Yang-Baxter equation with zero entries except in the places

$$\underline{R}^i_j{}^j{}_i = 1 \quad i, j = 1, 2, 3 \quad \underline{R}^1_2{}^3{}_2 = 1 - q \quad \underline{R}^2_3{}^3{}_2 = q^2 - q$$

where the left indices here numbered the block. Obviously this is a null degree solution with respect to the previous degree assignation $p(e_i)$. In the anyonic bialgebra (9)–(10) the generators a^i_i are of degree zero, a^1_3, a^2_1, a^3_2 are of degree 1 and a^1_2, a^2_3, a^3_1 are of degree 2 and satisfy the following relations: a^1_1 is central and

$$\begin{array}{lll}
a^3_3 a^3_2 = a^3_2 a^3_3 & a^3_3 a^3_1 = 0 & a^3_3 a^2_3 = q a^2_3 a^3_3 \\
a^3_3 a^2_2 = a^2_2 a^3_3 + (q-1) a^2_3 a^3_2 & a^3_3 a^2_1 = q^2 a^2_1 a^3_3 & a^3_3 a^1_3 = q^2 a^1_3 a^3_3 \\
a^3_3 a^1_2 = q a^1_2 a^3_3 + q(q-1) a^1_3 a^3_2 & a^3_2 a^3_1 = 0 & a^3_2 a^2_3 = a^2_3 a^3_2 \\
a^3_2 a^2_2 = a^2_2 a^3_2 + q(q-1) a^2_1 a^3_3 & a^3_2 a^2_1 = a^2_1 a^3_2 & a^3_2 a^1_3 = a^1_3 a^3_2 \\
a^3_2 a^1_2 = a^1_2 a^3_2 + (q^2-1) a^1_3 a^3_1 & a^3_1 a^3_3 = 0 & a^3_1 a^3_2 = 0 \\
(a^3_1)^2 = 0 & a^3_1 a^2_3 = q^2 a^2_3 a^3_1 & a^3_1 a^2_2 = a^2_2 a^3_1 \\
a^3_1 a^2_1 = 0 & a^3_1 a^1_3 = q a^1_3 a^3_1 & a^3_1 a^1_2 = q^2 a^1_2 a^3_1 \\
(a^2_3)^2 = q a^1_3 a^3_3 & a^2_3 a^2_2 = q a^2_2 a^3_3 & a^2_3 a^2_1 = 0 \\
a^2_3 a^1_3 = 0 & a^2_3 a^1_2 = q^2 a^1_2 a^3_3 & (a^2_2)^2 = a^1_1 a^3_3 + a^1_2 a^3_2 + q^2 a^1_3 a^3_1 \\
a^2_2 a^2_1 = 0 & a^2_2 a^1_3 = q a^1_3 a^2_2 & a^2_2 a^1_2 = q^2 a^1_2 a^2_2 + (1-q) a^1_3 a^2_1 \\
a^2_1 a^3_1 = 0 & a^2_1 a^2_3 = 0 & a^2_1 a^2_2 = 0 \\
(a^2_1)^2 = 0 & a^2_1 a^1_3 = q^2 a^1_3 a^2_1 & a^2_1 a^1_2 = q a^1_2 a^2_1 \\
a^1_3 a^2_3 = 0 & (a^1_3)^2 = 0 & a^1_3 a^1_2 = 0 \\
a^1_2 a^1_3 = 0 & (a^1_2)^2 = q a^1_1 a^1_3 &
\end{array}$$

together with the “strange” relations

$$\begin{array}{ll}
a^1_3 a^2_2 = -a^1_2 a^2_3 & a^2_2 a^3_1 = -q a^2_1 a^3_2 \\
a^2_3 a^3_1 = -q^2 a^2_1 a^3_3 & a^1_3 a^2_1 = -a^1_1 a^2_3 - q^2 a^1_2 a^2_2 \\
a^2_2 a^2_3 = -q a^1_2 a^3_3 - q^2 a^1_3 a^3_2.
\end{array}$$

From these formulae and the fact that q is a primitive root of unity it is simple to derive that $a^1_2, a^1_3, a^2_1, a^2_3, a^3_1$ have cube equal to zero and that the cube of a^2_2, a^3_2, a^3_3 are all central. It is interesting to realize that this algebra can be considerably reduced by quotienting it by the relations $a^1_1 = 1, a^2_1 = a^3_1 = 0$. These are all compatible with the coalgebra structure. After this reduction the anyonic bialgebra above can be expressed in terms of only three independent generators, namely $z_1 \equiv a^2_2, z_2 \equiv a^3_2, z_3 \equiv a^1_2$ of degree 0, 1, 2 respectively (the remaining generators being expressed in terms of them by $a^1_3 = q^2 z_3^2, a^2_3 = -q^2 z_3 z_1, a^3_3 = z_1^2 - z_3 z_2$ using the relations above). The quotient bialgebra is generated by these z_1, z_2, z_3 with algebra and coalgebra relations

$$z_2 \text{ central} \quad z_1 z_3 = q^2 z_3 z_1 \quad z_3^2 = 0 \quad (12)$$

$$\underline{\Delta} z_1 = z_1 \otimes z_1 - z_1 z_3 \otimes z_2$$

$$\begin{aligned}
\underline{\Delta} z_2 &= z_2 \otimes z_1 + z_1^2 \otimes z_2 - z_3 z_2 \otimes z_2 \\
\underline{\Delta} z_3 &= \mathbb{1} \otimes z_3 + z_3 \otimes z_1 + q^2 z_3^2 \otimes z_2 \\
\underline{\varepsilon}(z_1) &= 1, \quad \underline{\varepsilon}(z_2) = \underline{\varepsilon}(z_3) = 0.
\end{aligned} \tag{13}$$

Notice that the coalgebra stated implies that the central element z_1^3 is group-like. In this particular example it is possible to extend the anyonic bialgebra to an anyonic Hopf algebra just by assuming that the operator z_1^{-1} exists. With this condition the anyonic-antipode \underline{S} of each generator is well defined and given by

$$\underline{S}(z_1) = z_1^{-1} - z_1^{-3} z_2 z_3 \quad \underline{S}(z_2) = -z_1^{-3} z_2 \quad \underline{S}(z_3) = -z_3 z_1^{-1} - q z_1^{-3} z_2 z_3^2.$$

Any element of this graded Hopf algebra is expressed as a product on the generators and its antipode is calculated with the property $\underline{S}(ab) = q^{p(a)p(b)} \underline{S}(b) \underline{S}(a)$ in terms of antipode of the generators. We recall again here that $q = e^{2\pi i/3}$.

This example demonstrates a solution to one of the puzzles underlying any naive approach to the anyonic theory: we might well expect in many anyonic or \mathbf{Z}_n -graded algebras to have some generators with the n th power equal to zero. This could be viewed as a partial substitute for anyonic commutativity (just as super-commutativity implies that the square of an odd generator is zero). But the puzzle is how such cubic or higher order relations could come from an FRT-type construction where the relations are quadratic. We see now how this works in our example for $n = 3$ where, for example, a^1_2 has square equal to a product of two other generators one of which is zero when multiplied by another a^1_2 . Starting with quadratic relations even for arbitrary n is not an obstacle since the prescription to formulate anyonic bialgebras that we have presented here, in spite of its quadratic aspect, gives the right anyonic character to the bialgebra that it constructs.

4 Final remark

The material presented in this paper details one of the two graded bialgebras afforded by the modified FRT construction. The second one of universal enveloping algebra type will be fully studied in [5]. However, any of the two can be obtained from the other once the pairing between them is known so we anticipate the pairing here. Thus the anyonic bialgebra $\underline{A}(\underline{R})$ of the previous section is dually paired with another anyonic bialgebra $\underline{U}(\underline{R})$ with

generators $1, m_j^{\pm i}$, $i, j = 1, \dots, d$ and the pairing is given by

$$\langle m_j^{\pm i}, a_l^k \rangle = e^{\frac{2\pi i}{n} p(i)[p(k)-p(l)]} \underline{R}^{\pm i}_{j \cdot l}{}^k,$$

where \underline{R}^{\pm} indicates a solution of equation (6).

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